

Slant Riemannian maps from almost Hermitian manifolds

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Abstract. As a generalization of holomorphic submersions, anti-invariant submersions and slant submersions, we introduce slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We give examples, obtain the existence conditions of slant Riemannian maps and investigate harmonicity of such maps. We also obtain necessary and sufficient conditions for slant Riemannian maps to be totally geodesic and give a decomposition theorem for the total manifold.

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1. Introduction

Differentiable maps between Riemannian manifolds are important in differential geometry. There are certain types of differentiable maps between Riemannian manifolds whose existence influence the geometry of the source manifolds and the target manifolds. Differentiable maps between Riemannian manifolds are also useful to compare geometric structures defined on both manifolds. Basic maps in this manner are isometric immersions between Riemannian manifolds. Such maps are characterized by their Jacobian matrices and the induced metric which is symmetric positive definite bilinear form. The theory of isometric immersions is an active research area and it plays an important role in the development of modern differential geometry. The other basic maps for comparing geometric structures defined on Riemannian manifolds are Riemannian submersions and they were studied by O'Neill [17] and Gray [12]. The theory of Riemannian submersions is also a very active research field, for recent developments in this area see:[9].

In 1992, Fischer introduced Riemannian maps between Riemannian manifolds in [10] as a generalization of the notions of isometric immersions and Riemannian submersions. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between

Riemannian manifolds such that $0 < \text{rank} F < \min\{m, n\}$, where $\dim M_1 = m$ and $\dim M_2 = n$. Then we denote the kernel space of F_* by $\ker F_*$ and consider the orthogonal complementary space $\mathcal{H} = (\ker F_*)^\perp$ to $\ker F_*$ in TM_1 . Thus the tangent bundle of M_1 has the following decomposition

$$TM_1 = \ker F_* \oplus \mathcal{H}.$$

We denote the range of F_* by $\text{range} F_*$ and consider the orthogonal complementary space $(\text{range} F_*)^\perp$ to $\text{range} F_*$ in the tangent bundle TM_2 of M_2 . Since $\text{rank} F < \min\{m, n\}$, we always have $(\text{range} F_*)^\perp$. Thus the tangent bundle TM_2 of M_2 has the following decomposition

$$TM_2 = (\text{range} F_*) \oplus (\text{range} F_*)^\perp.$$

Now, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F_{*p_1}^h : (\ker F_{*p_1})^\perp \longrightarrow (\text{range} F_{*p_1})$ is a linear isometry between the inner product spaces $((\ker F_{*p_1})^\perp, g_1(p_1)|_{(\ker F_{*p_1})^\perp})$ and $(\text{range} F_{*p_1}, g_2(p_2)|_{\text{range} F_{*p_1}})$, $p_2 = F(p_1)$. Therefore Fischer stated in [10] that a Riemannian map is a map which is as isometric as it can be. In another words, F_* satisfies the equation

$$g_2(F_*X, F_*Y) = g_1(X, Y) \quad (1.1)$$

for X, Y vector fields tangent to \mathcal{H} . It follows that isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\text{range} F_*)^\perp = \{0\}$. It is known that a Riemannian map is a subimmersion [10] and this fact implies that the rank of the linear map $F_{*p} : T_p M_1 \longrightarrow T_{F(p)} M_2$ is constant for p in each connected component of M_1 , [1] and [10]. It is also important to note that Riemannian maps satisfy the eikonal equation which is a bridge between geometric optics and physical optics. For Riemannian maps and their applications in spacetime geometry, see: [11].

Let \bar{M} be a Kähler manifold with complex structure J and M a Riemannian manifold isometrically immersed in \bar{M} . A submanifold M is called holomorphic (complex) if $J(T_p M) \subset T_p M$, for every $p \in M$, where $T_p M$ denotes the tangent space to M at the point p . M is called totally real if $J(T_p M) \subset T_p M^\perp$ for every $p \in M$, where $T_p M^\perp$ denotes the normal space to M at the point p . On the other hand, a submanifold M is called slant if for all non-zero vector X tangent to M the angle $\theta(X)$ between JX and $T_p M$ is a constant, i.e, it does not depend on the choice of $p \in M$ and $X \in T_p M$ [5]. Holomorphic submanifolds and totally real submanifolds are slant submanifolds with $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold is called proper if it is neither holomorphic nor totally real.

Riemannian submersions between Riemannian manifolds equipped with differentiable structures were studied by Watson in [23]. As an analogue of holomorphic submanifolds, Watson defined almost Hermitian submersions between

almost Hermitian manifolds as follows: Let M be a complex m -dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J_M and N be a complex n -dimensional almost Hermitian manifold with Hermitian metric g_N and almost complex structure J_N . A Riemannian submersion $F : M \rightarrow N$ is called an almost Hermitian submersion if F is an almost complex mapping, i.e., $F_*J_M = J_NF_*$. The main result of this notion is that the vertical and horizontal distributions are J_M -invariant. Watson also showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases [23] and [9]. Since then almost Hermitian submersions have been extended to the almost contact manifolds [8], [14], locally conformal Kähler manifolds [15] and quaternion Kähler manifolds [13].

In [19], we introduced anti-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds as follows. Let M be a complex m -dimensional almost Hermitian manifold with Hermitian metric g_M and almost complex structure J and N be a Riemannian manifold with Riemannian metric g_N . Suppose that there exists a Riemannian submersion $F : M \rightarrow N$ such that the integral manifold of the distribution $\ker F_*$ is anti-invariant with respect to J , i.e., $J(\ker F_*) \subseteq (\ker F_*)^\perp$. Then we say that F is an anti-invariant Riemannian submersion. As a generalization of almost Hermitian submersions and anti-invariant Riemannian submersions, recently, we also introduced the notion of slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds [20] as follows: Let F be a Riemannian submersion from an almost Hermitian manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2) . If for any non-zero vector $X \in \Gamma(\ker F_*)$, the angle $\theta(X)$ between JX and the space $\ker F_*$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $\ker F_*$, then we say that F is a slant submersion. In this case, the angle θ is called the slant angle of the slant submersion.

In [22], as a generalization of almost Hermitian submersions, anti-invariant Riemannian submersions and slant submersions, we defined semi-invariant Riemannian maps from almost Hermitian manifolds and investigated the geometry of the total manifold and the base manifold by using the existence of such maps.

In this paper, as another generalization of Hermitian submersions, anti-invariant submersions and slant submersions, we define and study slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. In section 2, we recall basic facts for Riemannian maps and almost Hermitian manifolds. In section 3, we define slant Riemannian maps and give many examples. We also obtain a characterization of such maps and investigate the harmonicity of slant Riemannian maps. Then we give necessary and sufficient conditions for slant Riemannian maps to be totally geodesic. Finally, in section 4, we obtain a decomposition theorem for the total manifold by using slant Riemannian maps.

2. Riemannian maps

In this section, we develop fundamental formulas for Riemannian maps similar to the Gauss-Weingarten formulas of isometric immersions and O'Neill's formulas of Riemannian submersions. We also recall useful results which are related to the second fundamental form and the tension field of Riemannian maps. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \rightarrow N$ is a smooth map between them. Then the differential F_* of F can be viewed a section of the bundle $Hom(TM, F^{-1}TN) \rightarrow M$, where $F^{-1}TN$ is the pullback bundle which has fibres $(F^{-1}TN)_p = T_{F(p)}N, p \in M$. $Hom(TM, F^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) - F_*(\nabla_X^M Y) \quad (2.1)$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric [2]. First note that in [21] we showed that the second fundamental form $(\nabla F_*)(X, Y), \forall X, Y \in \Gamma((\ker F_*)^\perp)$, of a Riemannian map has no components in $range F_*$. More precisely we have the following.

Lemma 2.1. *Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then*

$$g_2((\nabla F_*)(X, Y), F_*(Z)) = 0, \forall X, Y, Z \in \Gamma((\ker F_*)^\perp).$$

As a result of Lemma 2.1, we have

$$(\nabla F_*)(X, Y) \in \Gamma((range F_*)^\perp), \forall X, Y \in \Gamma((\ker F_*)^\perp). \quad (2.2)$$

For the tension field of a Riemannian map between Riemannian manifolds, we have the following.

Lemma 2.2. [18] *Let $F : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian map between Riemannian manifolds. Then the tension field τ of F is*

$$\tau = -m_1 F_*(H) + m_2 H_2, \quad (2.3)$$

where $m_1 = \dim(\ker F_*)$, $m_2 = \text{rank } F$, H and H_2 are the mean curvature vector fields of the distribution $\ker F_*$ and $range F_*$, respectively.

Let F be a Riemannian map from a Riemannian manifold (M_1, g_1) to a Riemannian manifold (M_2, g_2) . Then we define \mathcal{T} and \mathcal{A} as

$$\mathcal{A}_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H} F \quad (2.4)$$

$$\mathcal{T}_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V} F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H} F, \quad (2.5)$$

for vector fields E, F on M_1 , where ∇ is the Levi-Civita connection of g_1 . In fact, one can see that these tensor fields are O'Neill's tensor fields which were defined

for Riemannian submersions. For any $E \in \Gamma(TM_1)$, \mathcal{T}_E and \mathcal{A}_E are skew-symmetric operators on $(\Gamma(TM_1), g)$ reversing the horizontal and the vertical distributions. It is also easy to see that \mathcal{T} is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and \mathcal{A} is horizontal, $\mathcal{A} = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field \mathcal{T} satisfies

$$\mathcal{T}_U W = \mathcal{T}_W U, \forall U, W \in \Gamma(\ker F_*). \quad (2.6)$$

On the other hand, from (2.4) and (2.5) we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W \quad (2.7)$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X \quad (2.8)$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V \quad (2.9)$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \quad (2.10)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

From now on, for simplicity, we denote by ∇^2 both the Levi-Civita connection of (M_2, g_2) and its pullback along F . Then according to [16], for any vector field X on M_1 and any section V of $(\text{range } F_*)^\perp$, where $(\text{range } F_*)^\perp$ is the subbundle of $F^{-1}(TM_2)$ with fiber $(F_*(T_p M))^\perp$ -orthogonal complement of $F_*(T_p M)$ for g_2 over p , we have $\nabla_X^{F^\perp} V$ which is the orthogonal projection of $\nabla_X^2 V$ on $(F_*(TM))^\perp$. In [16], the author also showed that ∇^{F^\perp} is a linear connection on $(F_*(TM))^\perp$ such that $\nabla^{F^\perp} g_2 = 0$. We now define \mathcal{S}_V as

$$\nabla_{F_* X}^2 V = -\mathcal{S}_V F_* X + \nabla_X^{F^\perp} V, \quad (2.11)$$

where $\mathcal{S}_V F_* X$ is the tangential component (a vector field along F) of $\nabla_{F_* X}^2 V$. It is easy to see that $\mathcal{S}_V F_* X$ is bilinear in V and $F_* X$ and $\mathcal{S}_V F_* X$ at p depends only on V_p and $F_{*p} X_p$. By direct computations, we obtain

$$g_2(\mathcal{S}_V F_* X, F_* Y) = g_2(V, (\nabla F_*)(X, Y)), \quad (2.12)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma((\text{range } F_*)^\perp)$. Since (∇F_*) is symmetric, it follows that \mathcal{S}_V is a symmetric linear transformation of $\text{range } F_*$.

A $2k$ -dimensional Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ is called an almost Hermitian manifold if there exists a tensor field \bar{J} of type (1,1) on \bar{M} such that $\bar{J}^2 = -I$ and

$$\bar{g}(X, Y) = \bar{g}(\bar{J}X, \bar{J}Y), \forall X, Y \in \Gamma(T\bar{M}), \quad (2.13)$$

where I denotes the identity transformation of $T_p \bar{M}$. Consider an almost Hermitian manifold $(\bar{M}, \bar{J}, \bar{g})$ and denote by $\bar{\nabla}$ the Levi-Civita connection on \bar{M} with respect to \bar{g} . Then \bar{M} is called a Kähler manifold if \bar{J} is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X \bar{J})Y = 0 \quad (2.14)$$

for $X, Y \in \Gamma(T\bar{M})$ [24].

3. Slant Riemannian maps

In this section, as a generalization of almost Hermitian submersions, slant submersions and anti-invariant Riemannian submersions, we introduce slant Riemannian maps from an almost Hermitian manifold to a Riemannian manifold. We first focus on the existence of such maps by giving some examples. Then we investigate the effect of slant Riemannian maps on the geometry of the total manifold, the base manifold and themselves. More precisely, we investigate the geometry of leaves of distributions on the total manifold arisen from such maps. We also obtain necessary and sufficient conditions for slant Riemannian maps to be harmonic and totally geodesic. We first present the following definition.

Definition 3.1. *Let F be a Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to a Riemannian manifold (M_2, g_2) . If for any non-zero vector $X \in \Gamma(\ker F_*)$, the angle $\theta(X)$ between JX and the space $\ker F_*$ is a constant, i.e. it is independent of the choice of the point $p \in M_1$ and choice of the tangent vector X in $\ker F_*$, then we say that F is a slant Riemannian map. In this case, the angle θ is called the slant angle of the slant Riemannian map.*

Since F is a subimmersion, it follows that the rank of F is constant on M_1 , then the rank theorem for functions implies that $\ker F_*$ is an integrable subbundle of TM_1 , ([1], page:205). Thus it follows from above definition that the leaves of the distribution $\ker F_*$ of a slant Riemannian map are slant submanifolds of M_1 , for slant submanifolds, see: [6].

We first give some examples of slant Riemannian maps.

Example 1. Every Hermitian submersion from an almost Hermitian manifold onto an almost Hermitian manifold is a slant Riemannian map with $\theta = 0$ and $(\text{range } F_*)^\perp = \{0\}$.

Example 2. Every anti-invariant Riemannian submersion from an almost Hermitian manifold onto a Riemannian manifold is a slant Riemannian map with $\theta = \frac{\pi}{2}$ and $(\text{range } F_*)^\perp = \{0\}$.

Example 3. Every proper slant submersion with the slant angle θ is a slant Riemannian map with $(\text{range } F_*)^\perp = \{0\}$.

We now denote the Euclidean $2m$ -space with the standard metric by R^{2m} . An almost complex structure J on R^{2m} is said to be compatible if (R^{2m}, J) is complex analytically isometric to the complex number space C^m with the standard flat Kählerian metric. Then the compatible almost complex structure J on R^{2m} defined by

$$J(a^1, \dots, a^{2m}) = (-a^{m+1}, -a^{m+2}, \dots, -a^{2m}, a^1, a^2, \dots, a^m).$$

A slant Riemannian map is said to be proper if it is not a submersion. Here

is an example of proper slant Riemannian maps.

Example 4. Consider the following Riemannian map given by

$$F : \begin{matrix} R^4 \\ (x_1, x_2, x_3, x_4) \end{matrix} \longrightarrow \begin{matrix} R^4 \\ (0, \frac{x_2 \sin \alpha + x_3 + x_4 \cos \alpha}{\sqrt{2}}, 0, x_2 \cos \alpha - x_4 \sin \alpha) \end{matrix}.$$

Then for any $0 < \alpha < \frac{\pi}{2}$, F is a slant Riemannian map with respect to the compatible almost complex structure J on R^4 with slant angle $\frac{\pi}{4}$.

Let F be a Riemannian map from a Kähler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then for $X \in \Gamma(\ker F_*)$, we write

$$JX = \phi X + \omega X, \quad (3.1)$$

where ϕX and ωX are vertical and horizontal parts of JX . Also for $V \in \Gamma((\ker F_*)^\perp)$, we have

$$JZ = \mathcal{B}Z + \mathcal{C}Z, \quad (3.2)$$

where $\mathcal{B}Z$ and $\mathcal{C}Z$ are vertical and horizontal components of JZ . Using (2.7), (2.8), (3.1) and (3.3) we obtain

$$(\nabla_X \omega)Y = \mathcal{C}\mathcal{T}_X Y - \mathcal{T}_X \phi Y \quad (3.3)$$

$$(\nabla_X \phi)Y = \mathcal{B}\mathcal{T}_X Y - \mathcal{T}_X \omega Y, \quad (3.4)$$

where ∇ is the Levi-Civita connection on M_1 and

$$\begin{aligned} (\nabla_X \omega)Y &= \mathcal{H}\nabla_X \omega Y - \omega \hat{\nabla}_X Y \\ (\nabla_X \phi)Y &= \hat{\nabla}_X \phi Y - \phi \hat{\nabla}_X Y \end{aligned}$$

for $X, Y \in \Gamma(\ker F_*)$. Let F be a slant Riemannian map from an almost Hermitian manifold (M_1, g_1, J_1) to a Riemannian manifold (M_2, g_2) , then we say that ω is parallel with respect to the Levi-Civita connection ∇ on $\ker F_*$ if its covariant derivative with respect to ∇ vanishes, i.e., we have

$$(\nabla_X \omega)Y = \nabla_X \omega Y - \omega(\nabla_X Y) = 0$$

for $X, Y \in \Gamma(\ker F_*)$. Let F be a slant Riemannian map from a complex m -dimensional Hermitian manifold (M, g_1, J) to a Riemannian manifold (N, g_2) . Then, $\omega(\ker F_*)$ is a subspace of $(\ker F_*)^\perp$. Thus it follows that $\ker F_{*p} \oplus \omega(\ker F_{*p})$ is invariant with respect to J . Then for every $p \in M$, there exists an invariant subspace μ_p of $(\ker F_{*p})^\perp$ such that

$$T_p M = \ker F_{*p} \oplus \omega(\ker F_{*p}) \oplus \mu_p.$$

The proof of the following result is exactly the same with slant immersions (see [5] or [3] and [4] for Sasakian case), therefore we omit its proof.

Theorem 3.1. *Let F be a Riemannian map from an almost Hermitian manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then F is a slant Riemannian map if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$\phi^2 X = \lambda X$$

for $X \in \Gamma(\ker F_*)$. If F is a slant Riemannian map, then $\lambda = -\cos^2 \theta$.

By using above theorem, it is easy to see that

$$g_1(\phi X, \phi Y) = \cos^2 \theta g_1(X, Y) \quad (3.5)$$

$$g_1(\omega X, \omega Y) = \sin^2 \theta g_1(X, Y) \quad (3.6)$$

for any $X, Y \in \Gamma(\ker F_*)$. Also by using (3.5) we can easily conclude that

$$\{e_1, \sec \theta \phi e_1, e_2, \sec \theta \phi e_2, \dots, e_n, \sec \theta \phi e_n\}$$

is an orthonormal frame for $\Gamma(\ker F_*)$. On the other hand, by using (3.6) one can see that

$$\{\csc \theta \omega e_1, \csc \theta \omega e_2, \dots, \csc \theta \omega e_n\}$$

is an orthonormal frame for $\Gamma(\omega(\ker F_*))$. As in slant immersions, we call the frame

$$\{e_1, \sec \theta \phi e_1, e_2, \sec \theta \phi e_2, \dots, e_n, \sec \theta \phi e_n, \csc \theta \omega e_1, \csc \theta \omega e_2, \dots, \csc \theta \omega e_n\}$$

an adapted frame for slant Riemannian maps.

We note that since the distribution $\ker F_*$ is integrable it follows that $\mathcal{T}_X Y = \mathcal{T}_Y X$ for $X, Y \in \Gamma(\ker F_*)$. Then the following Lemma can be obtained by using Theorem 3.1.

Lemma 3.1. *Let F be a slant Riemannian map from a Kähler manifold to a Riemannian manifold. If ω is parallel with respect to ∇ on $\ker F_*$, then*

$$\mathcal{T}_{\phi X} \phi X = -\cos^2 \theta \mathcal{T}_X X \quad (3.7)$$

for $X \in \Gamma(\ker F_*)$.

In fact, proof of the above Lemma is exactly the same with the Lemma 3.3 given in [20].

We now give necessary and sufficient conditions for F to be harmonic.

Theorem 3.2. *Let F be a slant Riemannian map from a Kähler manifold to a Riemannian manifold. Then F is harmonic if and only if*

$$\mathcal{T}_{\phi e_i} \phi e_i = -\cos^2 \theta \mathcal{T}_{e_i} e_i, \quad (3.8)$$

$$\text{trace } |_{\omega(\ker F_*)} {}^* F_*(\mathcal{S}_{E_j} F_*(\cdot)) \in \Gamma(\mu), \quad (3.9)$$

and

$$\text{trace } |_{\mu} {}^*F_*(\mathcal{S}_{E_j}F_*(\cdot)) \in \Gamma(\omega(\ker F_*)), \quad (3.10)$$

where $\{e_1, \sec \theta \phi e_1, e_2, \sec \theta \phi e_2, \dots, e_n, \sec \theta \phi e_n\}$ is an orthonormal frame for $\Gamma(\ker F_*)$ and $\{E_k\}$ is an orthonormal frame of $\Gamma((\text{range } F_*)^\perp)$.

Proof. We choose a canonical orthonormal frame $e_1, \sec \theta \phi e_1, \dots, e_p, \sec \theta \phi e_p, \omega \csc \theta e_1, \dots, \omega \csc \theta e_{2p}, \bar{e}_1, \dots, \bar{e}_n$ such that $\{e_1, \sec \theta \phi e_1, \dots, e_p, \sec \theta \phi e_p\}$ is an orthonormal basis of $\ker F_*$ and $\{\bar{e}_1, \dots, \bar{e}_n\}$ of μ , where θ is the slant angle. Then F is harmonic if and only if

$$\begin{aligned} \sum_{i=1}^p (\nabla F_*)(e_i, e_i) + \sec^2 \theta (\nabla F_*)(\phi e_i, \phi e_i) + \csc^2 \theta \sum_{i=1}^{2p} (\nabla F_*)(\omega e_i, \omega e_i) \\ + \sum_{j=1}^m (\nabla F_*)(\bar{e}_j, \bar{e}_j) = 0. \end{aligned} \quad (3.11)$$

By using (2.1) and (2.7) we have

$$\sum_{i=1}^p ((\nabla F_*)(e_i, e_i) + \sec^2 \theta (\nabla F_*)(\phi e_i, \phi e_i)) = -F_*(\mathcal{T}_{e_i} e_i + \sec^2 \theta \mathcal{T}_{\phi e_i} \phi e_i). \quad (3.12)$$

On the other hand from Lemma 2.1, we know $\csc^2 \theta \sum_{i=1}^{2p} (\nabla F_*)(\omega e_i, \omega e_i) + \sum_{j=1}^m (\nabla F_*)(\bar{e}_j, \bar{e}_j) \in \Gamma((\text{range } F_*)^\perp)$. Thus we can write

$$\begin{aligned} \csc^2 \theta \sum_{i=1}^{2p} (\nabla F_*)(\omega e_i, \omega e_i) + \sum_{j=1}^m (\nabla F_*)(\bar{e}_j, \bar{e}_j) &= \csc^2 \theta \\ \sum_{i=1}^{2p} \sum_{k=1}^s g_2((\nabla F_*)(\omega e_i, \omega e_i), E_k) E_k \\ + \sum_{j=1}^m \sum_{k=1}^s g_2((\nabla F_*)(\bar{e}_j, \bar{e}_j), E_k) E_k \end{aligned}$$

where $\{E_k\}$ is an orthonormal basis of $\Gamma((\text{range } F_*)^\perp)$. Then using (2.12) we have

$$\begin{aligned} \csc^2 \theta \sum_{i=1}^{2p} (\nabla F_*)(\omega e_i, \omega e_i) + \sum_{j=1}^m (\nabla F_*)(\bar{e}_j, \bar{e}_j) &= \csc^2 \theta \\ \sum_{i=1}^{2p} \sum_{k=1}^s g_2(\mathcal{S}_{E_k} F_*(\omega e_i), F_*(\omega e_i)) E_k \\ + \sum_{j=1}^m \sum_{k=1}^s g_2(\mathcal{S}_{E_k} F_*(\bar{e}_j), F_*(\bar{e}_j)) E_k \end{aligned} \quad (3.13)$$

Then proof comes from the adjoint of F_* , (3.12) and (3.13).

Example 5. Consider the slant Riemannian map given in Example 4, then we have

$$(\ker F_*) = \text{Span}\{Z_1 = \frac{\partial}{\partial x_1}, Z_2 = \sin \alpha \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \cos \alpha \frac{\partial}{\partial x_4}\}$$

and

$$\begin{aligned} (\ker F_*)^\perp = \text{Span}\{Z_3 &= \frac{\sin \alpha}{\sqrt{2}} \frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_3} + \frac{\cos \alpha}{\sqrt{2}} \frac{\partial}{\partial x_4}, \\ Z_4 &= -\cos \alpha \frac{\partial}{\partial x_2} + \sin \alpha \frac{\partial}{\partial x_4}\}. \end{aligned}$$

By direct computations, we have

$$JZ_1 = -\frac{1}{2}Z_2 + \frac{1}{\sqrt{2}}Z_3, JZ_2 = Z_1 + Z_4$$

which imply that

$$\phi Z_1 = -\frac{1}{2}Z_2, \phi Z_2 = Z_1.$$

Then it is easy to see that

$$\phi^2 Z_i = -\cos^2 \frac{\pi}{4} Z_i = -\frac{1}{2} Z_i, i = 1, 2$$

which is the statement of Theorem 3.1. On the other hand, since \mathcal{T} and \mathcal{S} vanish for this slant Riemannian map, it satisfies the claim of Theorem 3.2.

By using (2.7) and (3.3), one can notice that the equality (3.8) is satisfied in terms of the tensor field ω . More precisely, we have the following.

Lemma 3.2. *Let F be a slant Riemannian map from a Kähler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . If ω is parallel then (3.8) is satisfied.*

Remark 1. We note that the equality (3.7) (as a result of above lemma, parallel ω) is enough for a slant submersion to be harmonic, however for a slant Riemannian map this case is not valid anymore.

We now investigate necessary and sufficient conditions for a slant Riemannian map F to be totally geodesic. We recall that a differentiable map F between Riemannian manifolds (M_1, g_1) and (M_2, g_2) is called a totally geodesic map if $(\nabla F_*)(X, Y) = 0$ for all $X, Y \in \Gamma(TM_1)$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths.

Theorem 3.3. *Let F be a slant Riemannian map from a Kähler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then F is totally geodesic if and only if*

$$g_1(\mathcal{T}_U \omega V, \mathcal{B}X) = -g_2((\nabla F_*)(U, \omega \phi V), F_*(X)) + g_2((\nabla F_*)(U, \omega V), F_*(\mathcal{C}X))$$

$$g_1(\mathcal{A}_X \omega U, \mathcal{B}Y) = g_2(\nabla_X^F F_*(\omega \phi U), F_*(Y)) - g_2(\nabla_X^F F_*(\omega U), F_*(\mathcal{C}Y))$$

and

$$\nabla_X^F F_*(Y) + F_*(\mathcal{C}(\mathcal{A}_X \mathcal{B}Y + \mathcal{H} \nabla_X^1 \mathcal{C}Y) + \omega(\mathcal{V} \nabla_X^1 \mathcal{B}Y + \mathcal{A}_X \mathcal{C}Y)) \in \Gamma(\text{range } F_*)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$, where ∇^1 is the Levi-Civita connection of M_1 .

Proof. From the decomposition of the total manifold of a slant Riemannian map, it follows that F is totally geodesic if and only if $g_2((\nabla F_*)(U, V), F_*(X)) = 0$, $g_2((\nabla F_*)(X, U), F_*(Y)) = 0$ and $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$. First, since F is a Riemannian map, from (2.1) we obtain

$$g_2((\nabla F_*)(U, V), F_*(X)) = -g_1(\nabla_U^1 V, X).$$

Since M_1 is a Kähler manifold, using (3.1) and (3.2) we have

$$\begin{aligned} g_2((\nabla F_*)(U, V), F_*(X)) &= -\cos^2 \theta g_1(\nabla_U^1 V, X) + g_1(\nabla_U^1 \omega \phi V, X) \\ &\quad - g_1(\nabla_U^1 \omega V, \mathcal{B}X) - g_1(\nabla_U^1 \omega V, \mathcal{C}X). \end{aligned}$$

Taking into account that F is a Riemannian map, using again (2.1) and (2.8) we get

$$\begin{aligned} g_2((\nabla F_*)(U, V), F_*(X)) &= \sec^2 \theta \{-g_1(\mathcal{T}_U \omega V, \mathcal{B}X) - g_2((\nabla F_*)(U, \omega \phi V), F_*(X)) \\ &\quad + g_2((\nabla F_*)(U, \omega V), F_*(\mathcal{C}X))\}. \end{aligned} \quad (3.14)$$

In a similar way, we also have

$$\begin{aligned} g_2((\nabla F_*)(X, U), F_*(Y)) &= \sec^2 \theta \{-g_1(\mathcal{A}_X \omega U, \mathcal{B}Y) - g_2(\nabla_X^F F_*(\omega U), F_*(\mathcal{C}Y)) \\ &\quad + g_2(\nabla_X^F F_*(\omega \phi U), F_*(Y))\}. \end{aligned} \quad (3.15)$$

On the other hand, by using (2.1) and (2.14) we derive

$$(\nabla F_*)(X, Y) = \nabla_X^F F_*(Y) + F_*(J \nabla_X^1 JY)$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. Then using (3.1), (3.2) and (2.7)-(2.13) we obtain

$$\begin{aligned} (\nabla F_*)(X, Y) &= \nabla_X^F F_*(Y) + F_*(\mathcal{B} \mathcal{A}_X \mathcal{B}Y \\ &\quad + \mathcal{C} \mathcal{A}_X \mathcal{B}Y + \phi \mathcal{V} \nabla_X^1 \mathcal{B}Y + \omega \mathcal{V} \nabla_X^1 \mathcal{B}Y \\ &\quad + \mathcal{B} \mathcal{H} \nabla_X^1 \mathcal{C}Y + \mathcal{C} \mathcal{H} \nabla_X^1 \mathcal{C}Y \\ &\quad + \phi \mathcal{A}_X \mathcal{C}Y + \omega \mathcal{A}_X \mathcal{C}Y). \end{aligned}$$

Since

$$\mathcal{B}\mathcal{A}_X\mathcal{B}Y + \phi\mathcal{V}\nabla_X^1\mathcal{B}Y + \mathcal{B}\mathcal{H}\nabla_X^1\mathcal{C}Y + \phi\mathcal{A}_X\mathcal{C}Y \in \Gamma(\ker F_*),$$

we have

$$\begin{aligned} (\nabla F_*)(X, Y) &= \nabla_X^F F_*(Y) + F_*(\mathcal{C}\mathcal{A}_X\mathcal{B}Y \\ &\quad + \omega\mathcal{V}\nabla_X^1\mathcal{B}Y + \mathcal{C}\mathcal{H}\nabla_X^1\mathcal{C}Y \\ &\quad + \omega\mathcal{A}_X\mathcal{C}Y). \end{aligned} \quad (3.16)$$

Then proof comes from (3.14), (3.15) and (3.16).

Remark 2. Since \mathcal{T} , \mathcal{A} and (∇F_*) vanish, Example 5 satisfies the conditions of Theorem 3.3.

Remark 3. We observe that the conditions for a slant Riemannian map to be a totally geodesic are different from the conditions for a slant submersion to be totally geodesic, compare Theorem 3.3 of the present paper with Theorem 3.5 of [20]. For a Riemannian submersion, the second fundamental form satisfies $(\nabla F_*)(X, Y) = 0$, $X, Y \in \Gamma((\ker F_*)^\perp)$. However, for a slant Riemannian map there is no guarantee that $(\nabla F_*)(X, Y) = 0$, $X, Y \in \Gamma((\ker F_*)^\perp)$. From Lemma 2.1, we only know that $(\nabla F_*)(X, Y)$ is $\Gamma((\text{range } F_*)^\perp)$ -valued. From the above reason it is necessary to use extra geometric conditions to investigate the geometry of slant Riemannian maps.

4. A decomposition theorem via slant Riemannian maps

In this section we are going to obtain necessary and sufficient conditions for the total manifold of a slant Riemannian map to be a locally product Riemannian manifold. Let g be a Riemannian metric tensor on the manifold $M = B \times F$ and assume that the canonical foliations D and \bar{D} intersect perpendicularly everywhere. Then from de Rham's theorem [7], we know that g is the metric tensor of a usual product Riemannian manifold if and only if D and \bar{D} are totally geodesic foliations.

Theorem 4.1. *Let F be a slant Riemannian map from a Kähler manifold (M_1, g_1, J) to a Riemannian manifold (M_2, g_2) . Then (M_1, g_1) is a locally product Riemannian manifold if and only if*

$$g_1(\mathcal{T}_U\omega V, \mathcal{B}X) = -g_2((\nabla F_*)(U, \omega\phi V), F_*(X)) + g_2((\nabla F_*)(U, \omega V), F_*(\mathcal{C}X))$$

and

$$g_2((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)) = g_2(F_*(Y), \nabla_X^F F_*(\omega\phi U)) - g_2(F_*(\mathcal{C}Y), \nabla_X^F F_*(\omega U))$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$.

Proof. For $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$, from (2.14), (3.1), (3.2) and Theorem 3.1, we have

$$\begin{aligned} g_1(\nabla_X^1 Y, U) &= -\cos^2 \theta g_1(Y, \nabla_X^1 U) + g_1(Y, \nabla_X^1 \omega \phi U) \\ &\quad - g_1(\mathcal{B}Y, \nabla_X^1 \omega U) - g_1(CY, \nabla_X^1 \omega U). \end{aligned}$$

Taking into account that F is a Riemannian map and using (2.1) we obtain

$$\begin{aligned} g_1(\nabla_X^1 Y, U) &= \sec^2 \theta \{-g_2(F_*(Y), (\nabla F_*)(X, \omega \phi U)) + g_2(F_*(Y), \nabla_X^F F_*(\omega \phi U)) \\ &\quad - g_2((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)) + g_2((\nabla F_*)(X, \omega U), F_*(CY)) \\ &\quad - g_2(F_*(CY), \nabla_X^F F_*(\omega U))\}. \end{aligned}$$

Then Lemma 2.1 implies that

$$\begin{aligned} g_1(\nabla_X^1 Y, U) &= \sec^2 \theta \{g_2(F_*(Y), \nabla_X^F F_*(\omega \phi U)) - g_2((\nabla F_*)(X, \mathcal{B}Y), F_*(\omega U)) \\ &\quad - g_2(F_*(CY), \nabla_X^F F_*(\omega U))\}. \end{aligned} \quad (4.1)$$

Thus proof follows from (3.14) and (4.1).

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